

# Two-photon interaction between trapped ions and cavity fields

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**Abstract.** In this paper, we generalize the ordinary two-photon Jaynes-Cummings model (TPJCM) by considering the atom (or ion) to be trapped in a simple harmonic well. A typical setup would be an optical cavity containing a single ion in a Paul trap. Due to the inclusion of atomic vibrational motion, the atom-field coupling becomes highly nonlinear what brings out quite different behaviours for the system dynamics when compared to the ordinary TPJCM. In particular, we derive an effective two-photon Hamiltonian with dependence on the number operator of the ion’s center-of-mass motion. This dependence occurs both in the cavity induced Stark-shifts and in the ion-field coupling, and its role in the dynamics is illustrated by showing the time evolution of the probability of occupation of the electronic levels for simple initial preparations of the state of the system.

**PACS.** 42.50.-p Quantum optics – 32.80.Qk Coherent control of atomic interactions with photons – 32.80.Wr Other multiphoton processes

## 1 Introduction

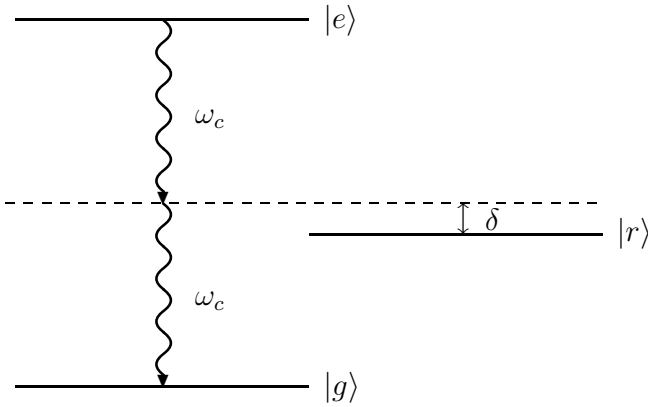
The Jaynes-Cummings model (JCM) [1] is the most elementary quantum model for the interaction of matter with an electromagnetic field. In this model, a two-level atom is coupled to a single quantized mode of a cavity field and it is one of the few fully quantum-mechanical models that is exactly solvable. It exhibits some unexpected nonclassical features as the revivals of the Rabi oscillations in the atomic inversion, for instance [2]. The quantum origin of this revival is a direct consequence of the non-vanishing commutation relation between the creation and annihilation operator of photons in the field mode [2]. During the past decades, the JCM has been extended to more general Hamiltonians including multilevel atoms [3], multimode or external fields [4], multi-atom configurations [5], and multi-photon transitions [6], just to mention a few examples. All these generalizations are part of what is called cavity quantum electrodynamics (cavity QED) and the quantized electromagnetic field plays a fundamental role in those settings. By the other hand, trapped ions interacting with classical fields have gained considerable interest in the past few years, mainly because the system dynamics closely resembles that of the JCM, with the quantized harmonic motion of the ion’s center-of-mass playing the role of the field [7]. Significant experimental advances in the generation of quantum states in such a system [8] have also indicated that trapped ions are very suited for the study and physical implementation of quantum dynamics [9]. It seems clear that a system comprising quantum

cavity fields and trapped ions undergoing quantized harmonic motion could bring many interesting consequences and potential applications. A typical setup would be that in which an ion trap is inserted in a high finesse optical resonator so that the ion could interact with both quantized and classical external fields [10]. Several authors have studied this new setting in the framework of single-photon transitions [11] and also Raman transitions driven by the cavity field and a classical external laser [12, 13]. However, we are not restricted to single-photon transitions in the realm of cavity QED. Other processes, such as two-photon transitions, are particularly important and have potential applications, e.g., in the generation of non-classical states of the electromagnetic field, such as squeezed states [14] or even photons with correlations that violate classical inequalities [15]. We would like to point out that several nonlinear effects in trapped ions interacting with classical laser fields have also been studied and applications suggested [16, 17].

This paper is concerned with the study of atom-field two-photon interactions when including harmonic atomic motion. We derive an effective Hamiltonian which not only describes electronic two-photon transitions but also contains a kind of phase-coupling between the vibrational center-of-mass motion, the electronic degree of freedom, and the cavity field. Another interesting feature of that Hamiltonian is the presence of Stark-shifts depending on the number operator of the harmonic motion. We then analyze the system dynamics by means of the probability of finding the ion in the electronic ground state, which is an easily accessible quantity in ion trap experiments [8].

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**Fig. 1.** Schematic diagram of the three-level ion interacting with a single mode quantized field with frequency  $\omega_c$ .

We discuss how this quantity is influenced by the statistics of simple initial preparations of the quantum state of motion. In other words, we systematically compare our model with the important and extensively studied TPJCM without the motional effects [6].

## 2 The model and results

We consider a three-level ion in a cascade configuration interacting with a single-mode quantized electromagnetic field enclosed in a high finesse cavity. The schematic level structure is depicted in Figure 1. It is assumed two-photon resonance between the upper  $|e\rangle$  and lower  $|g\rangle$  atomic levels and the intermediate level  $|r\rangle$  is kept off-resonant. The general Hamiltonian describing the interaction of trapped ions (or atoms) and electromagnetic fields is discussed in several papers [7, 10–13, 16, 17], and in our case it reads

$$\hat{H} = \nu \hat{a}^\dagger \hat{a} + \omega \hat{b}^\dagger \hat{b} + E_e \hat{\sigma}_{ee} + E_r \hat{\sigma}_{rr} + E_g \hat{\sigma}_{gg} + [g_1 (\hat{\sigma}_{gr} \hat{b}^\dagger + \hat{\sigma}_{rg} \hat{b}) + g_2 (\hat{\sigma}_{re} \hat{b}^\dagger + \hat{\sigma}_{er} \hat{b})] \cos \eta (\hat{a}^\dagger + \hat{a}) \quad (1)$$

where  $\hat{a}^\dagger(\hat{a})$  denotes the creation (annihilation) operator of the center-of-mass vibrational motion of the ion (frequency  $\nu$ ),  $\hat{b}^\dagger(\hat{b})$  is the creation (annihilation) operator of photons in the field mode (frequency  $\omega$ ),  $\hat{\sigma}_{ij} = |i\rangle\langle j|$  is a transition atomic operator,  $E_i$  is the energy of the atomic level  $|i\rangle$ ,  $g_1$  e  $g_2$  are the ion-field coupling constants for the transitions  $|g\rangle \rightarrow |r\rangle$  and  $|r\rangle \rightarrow |e\rangle$ , respectively, and  $\eta = 2\pi a_0/\lambda$  is the Lamb-Dicke (LD) parameter, being  $a_0$  the rms fluctuation of the ion's position in the lowest trap eigenstate, and  $\lambda$  the wavelength of the cavity field. We have taken ( $\hbar = 1$ ) and this convention will be followed in the rest of our paper. The Lamb-Dicke parameter is the same for both transitions (coupling constants  $g_1$  and  $g_2$ ) because both are driven by the same (cavity) field, i.e. with the same frequency (or  $\mathbf{k}$  vector). The cascade level structure and the field frequency [see Fig. 1] have been chosen such that by making

$\delta = E_e - E_r - \omega = E_g - E_r + \omega \gg g_1, g_2$  one can derive an effective two-photon Hamiltonian by means of the adiabatic elimination of the level  $|r\rangle$ .

The general Hamiltonian (1) is highly nonlinear because the function  $\cos \eta (\hat{a}^\dagger + \hat{a})$  contains powers of operators of the center-of-mass motion. Each term in the power series expansion will be dependent on powers of the LD parameter  $\eta$ . We can say generally that the higher the value of  $\eta$  the stronger will be the influence of nonlinear terms in the Hamiltonian (1). It is well-know that appropriate choices of the ion-field detuning  $\delta$  may lead to different kinds of couplings in the rotating wave approximation (RWA) [7]. In general, it is assumed  $\delta = k\nu$ , with  $k$  integer, what leads to either transitions between the trap eigenstates (k-sideband Hamiltonian, for  $k \neq 0$ ) or just a phase-coupling with no energy transitions for the ion's center-of-mass harmonic motion (carrier Hamiltonian, for  $k = 0$ ). However, it is not necessary to have  $\delta = 0$  in order to forbid those transitions (make them unlikely). The same result applies for a less demanding situation in which  $\delta \neq k\nu$  with  $\delta \ll \nu$ . The advantage of having this less demanding condition on the detuning  $\delta$  is that now it is possible to think of a situation where the field and atom can be kept far off-resonance ( $\delta \gg g_1, g_2$ ) without having excitations of the center-of-mass motion. This may be achieved by setting  $\nu \gg \delta \gg g_1, g_2$  and, as stated before,  $\delta \neq k\nu$ . In this case, it is possible to obtain a carrier two-photon Hamiltonian as we are going to show next.

The Hamiltonian (1) in the interaction picture reads

$$\hat{H}_I = [g_1 (\hat{\sigma}_{rg} \hat{b} e^{-i\delta t} + \hat{\sigma}_{gr} \hat{b}^\dagger e^{i\delta t}) + g_2 (\hat{\sigma}_{er} \hat{b} e^{i\delta t} + \hat{\sigma}_{re} \hat{b}^\dagger e^{-i\delta t})] \times \cos \eta (\hat{a}^\dagger e^{i\nu t} + \hat{a} e^{-i\nu t}), \quad (2)$$

which can be rewritten as

$$\hat{H}_I = \sum_{\alpha, \beta} [g_1 (\hat{\sigma}_{rg} \hat{b} e^{-i[\delta + \nu(\alpha - \beta)]t} + \hat{\sigma}_{gr} \hat{b}^\dagger e^{i[\delta + \nu(\alpha - \beta)]t}) + g_2 (\hat{\sigma}_{er} \hat{b} e^{i[\delta + \nu(\alpha - \beta)]t} + \hat{\sigma}_{re} \hat{b}^\dagger e^{-i[\delta + \nu(\alpha - \beta)]t})] \times f(\hat{a}^\dagger, \hat{a}; \alpha, \beta), \quad (3)$$

where

$$f(\hat{a}^\dagger, \hat{a}; \alpha, \beta) = \frac{e^{-\eta^2/2}}{2 \alpha! \beta!} [(i\eta)^{\alpha + \beta} + (-i\eta)^{\alpha + \beta}] a^\dagger{}^\alpha a^\beta. \quad (4)$$

Analyzing the temporal dependence of Hamiltonian (3), one can see that the frequencies may be carefully chosen allowing the RWA to be performed. In this approximation, only the slow frequency terms are kept while the rapidly oscillating ones are discarded. For sufficiently short interaction times, this approximation is quite accurate as far as the coupling constants are not too strong. How strong the coupling constants must be when compared to the other frequencies of the problem is usually found by using time dependent perturbation theory. In our problem, the terms in (3) oscillate in time as  $e^{i\pm(\delta + k\nu)t}$ , with  $k$  integer. In the regime  $\delta \ll \nu$  and  $\delta \neq k\nu$ , the slowly oscillating terms are

those with temporal dependence  $e^{\pm i\delta t}$ . This only happens when  $\alpha = \beta$  in (3). Then, by dropping out the rapidly oscillating terms in (3), we arrive at the following approximate Hamiltonian in the original picture

$$\begin{aligned} \hat{H} = & \nu \hat{a}^\dagger \hat{a} + \omega \hat{b}^\dagger \hat{b} + E_e \hat{\sigma}_{ee} + E_r \hat{\sigma}_{rr} + E_g \hat{\sigma}_{gg} \\ & + f(\hat{a}^\dagger \hat{a}) [(g_1 \hat{\sigma}_{gr} + g_2 \hat{\sigma}_{re}) \hat{b}^\dagger + (g_1 \hat{\sigma}_{rg} + g_2 \hat{\sigma}_{er}) \hat{b}] \end{aligned} \quad (5)$$

where [17,18],

$$\begin{aligned} f(\hat{a}^\dagger \hat{a}) &= e^{-\eta^2/2} \sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha \eta^{2\alpha} a^\dagger{}^\alpha a^\alpha}{\alpha!^2} \\ &= e^{-\eta^2/2} : J_0(2\eta\sqrt{\hat{a}^\dagger \hat{a}}) :, \end{aligned} \quad (6)$$

and  $: J_0 :$  is the normally ordered zeroth order Bessel function of the first kind. In an appendix in [18], it is shown how to convert  $: J_0 :$  in a form that does not contain the normal ordering symbol but is an expansion in Fock basis. This is very useful for doing the plots presented in the last part of this paper, and the relation between both forms is in our case given by

$$: J_0(2\eta\sqrt{\hat{a}^\dagger \hat{a}}) := \sum_{n=0}^{\infty} L_n(\eta^2) |n\rangle \langle n|, \quad (7)$$

where  $L_n$  is the Laguerre polynomial. The Hamiltonian (5) describes a situation in which the ion's center-of-mass motion couples to field and electronic degree of freedom in such a way that only phases are involved, i.e. there is no real transitions between vibrational energy levels. There is just photonic transitions taking place. The magnitude of this phase-coupling is contained in  $f(\hat{a}^\dagger \hat{a})$  which is a function of the number operator of the vibrational motion. In the limit  $\eta \rightarrow 0$  the function  $f(\hat{a}^\dagger \hat{a})$  tends to the identity operator, what corresponds to the free motion of the ion.

In order to derive an effective two-photon Hamiltonian, let us now find the equations of motion for some relevant operators. The starting point would be  $\hat{\sigma}_{eg}$  that is supposed to be present in the effective Hamiltonian once its function is to cause the ion to make a direct two-photon transition from the state  $|g\rangle$  to  $|e\rangle$ .

The Heisenberg equation for  $\hat{\sigma}_{eg}$  using (5) is given by

$$i \frac{d}{dt} \hat{\sigma}_{eg} = (E_g - E_e) \hat{\sigma}_{eg} + g_1 f(\hat{a}^\dagger \hat{a}) \hat{\sigma}_{er} \hat{b}^\dagger - g_2 f(\hat{a}^\dagger \hat{a}) \hat{\sigma}_{rg} \hat{b}^\dagger. \quad (8)$$

The right hand side of (8) involves operators in the form  $f(\hat{a}^\dagger \hat{a}) \hat{\sigma}_{ij} \hat{b}^\dagger$ . We need to compute the Heisenberg equations for these operators as well, and they are given by

$$\begin{aligned} i \frac{d}{dt} [f(\hat{a}^\dagger \hat{a}) \hat{\sigma}_{er} \hat{b}^\dagger] &= (E_r - E_e - \omega_c) f(\hat{a}^\dagger \hat{a}) \hat{\sigma}_{er} \hat{b}^\dagger \\ &+ [f(\hat{a}^\dagger \hat{a})]^2 [g_1 \hat{b}^\dagger \hat{b} \hat{\sigma}_{eg} + g_2 \hat{b}^{\dagger 2} (\hat{\sigma}_{ee} - \hat{\sigma}_{rr})] \end{aligned} \quad (9)$$

$$\begin{aligned} i \frac{d}{dt} [f(\hat{a}^\dagger \hat{a}) \hat{\sigma}_{rg} \hat{b}^\dagger] &= (E_g - E_r - \omega_c) f(\hat{a}^\dagger \hat{a}) \hat{\sigma}_{rg} \hat{b}^\dagger \\ &+ [f(\hat{a}^\dagger \hat{a})]^2 [g_1 \hat{b}^{\dagger 2} (\hat{\sigma}_{rr} - \hat{\sigma}_{gg}) \\ &- g_2 (1 + \hat{b}^\dagger \hat{b}) \hat{\sigma}_{eg}]. \end{aligned} \quad (10)$$

The adiabatic elimination of the level  $|r\rangle$  follows from (9) and (10) by considering the condition  $\delta \gg g_1, g_2$ . To make this clear, it is convenient to define new operators in an appropriate reference frame as  $\hat{\sigma}_{ij} \rightarrow \hat{\sigma}_{ij} e^{i(E_j - E_i)t}$ ,  $\hat{b}(t) \rightarrow \hat{b} e^{-i\omega_c t}$  e  $\hat{a}(t) \rightarrow \hat{a} e^{-i\nu t}$ . In this new frame, we may rewrite (8), (9) and (10) as

$$i \frac{d}{dt} \hat{\sigma}_{eg} = g_1 f(\hat{a}^\dagger \hat{a}) \hat{\sigma}_{er} \hat{b}^\dagger e^{i\delta t} + g_2 f(\hat{a}^\dagger \hat{a}) \hat{\sigma}_{rg} \hat{b}^\dagger e^{-i\delta t}, \quad (11)$$

$$\begin{aligned} i \frac{d}{dt} [f(\hat{a}^\dagger \hat{a}) \hat{\sigma}_{er} \hat{b}^\dagger] &= [f(\hat{a}^\dagger \hat{a})]^2 [g_1 \hat{b}^\dagger \hat{b} \hat{\sigma}_{eg} \\ &+ g_2 \hat{b}^{\dagger 2} (\hat{\sigma}_{ee} - \hat{\sigma}_{rr})] e^{-i\delta t}, \end{aligned} \quad (12)$$

$$\begin{aligned} i \frac{d}{dt} [f(\hat{a}^\dagger \hat{a}) \hat{\sigma}_{rg} \hat{b}^\dagger] &= [f(\hat{a}^\dagger \hat{a})]^2 [g_1 \hat{b}^{\dagger 2} (\hat{\sigma}_{rr} - \hat{\sigma}_{gg}) \\ &- g_2 (1 + \hat{b}^\dagger \hat{b}) \hat{\sigma}_{eg}] e^{i\delta t}. \end{aligned} \quad (13)$$

We then integrate (12) and (13) under the assumption that  $\delta \gg g_1, g_2$ . By substituting the result of the integrations in (11) and also considering that the level  $|r\rangle$  is not initially populated, we end up with the following equation of motion in the original Heisenberg picture

$$\begin{aligned} i \frac{d}{dt} \hat{\sigma}_{eg} &= f^2(\hat{a}^\dagger \hat{a}) \left[ \frac{g_1^2}{\delta} \hat{b}^\dagger \hat{b} - \frac{g_2^2}{\delta} (1 + \hat{b}^\dagger \hat{b}) \right] \hat{\sigma}_{eg} \\ &+ \frac{g_1 g_2}{\delta} f^2(\hat{a}^\dagger \hat{a}) (\hat{\sigma}_{ee} - \hat{\sigma}_{gg}) \hat{b}^{\dagger 2} \end{aligned} \quad (14)$$

which may be obtained from the effective Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{Stark}} + \hat{H}_I, \quad (15)$$

being the free part given by

$$\hat{H}_0 = \nu \hat{a}^\dagger \hat{a} + \omega_c \hat{b}^\dagger \hat{b} + \omega_c (\hat{\sigma}_{ee} - \hat{\sigma}_{gg}), \quad (16)$$

the Stark shifts

$$\hat{H}_{\text{Stark}} = \frac{g_2^2}{\delta} f^2(\hat{a}^\dagger \hat{a}) (1 + \hat{b}^\dagger \hat{b}) \hat{\sigma}_{ee} + \frac{g_1^2}{\delta} f^2(\hat{a}^\dagger \hat{a}) \hat{b}^\dagger \hat{b} \hat{\sigma}_{gg}, \quad (17)$$

and the two-photon interaction term between ion and field given by

$$\hat{H}_I = \frac{g_1 g_2}{\delta} f^2(\hat{a}^\dagger \hat{a}) (\hat{\sigma}_{eg} \hat{b}^2 + \hat{\sigma}_{ge} \hat{b}^{\dagger 2}). \quad (18)$$

It describes two-photon transitions between the levels  $|e\rangle$  and  $|g\rangle$  with a coupling constant that depends on energy of the center-of-mass motion via  $f^2(\hat{a}^\dagger \hat{a})$ .

An interesting feature of the Hamiltonian (15) is the dependence of the Stark shifts upon the motion, fact that is mathematically expressed by the presence of  $f^2(\hat{a}^\dagger \hat{a})$  in (17). This is the first remarkable feature of our model that is not present in the ordinary TPJCM. This means that the vibrational degree of freedom will have an important influence on the dynamics of the system not only due to the statistics of the quantum state of motion and the phase-coupling with the rest of the system but also via the Stark-shifts. As we are going to see below, even small variations of the Lamb-Dicke parameter will produce significant changes in the dynamics of the system. This makes

the model here presented useful for the investigation of quantum aspects of the light-matter interaction.

Once we have derived the phase-coupling two-photon Hamiltonian with motion-dependent Stark-shifts (15), which is the main result of our paper, we now proceed with the study of the dynamics of the electronic levels for simple but interesting and experimentally feasible initial preparations of the system. In order to do that, we will compute the occupation of level  $|g\rangle$  that is defined as

$$P_g(t) = |\langle g|\psi(t)\rangle|^2. \quad (19)$$

It follows from the form of (15) that as long as the ion is initially in its electronic excited state, the global state of the system may be written as

$$|\psi(t)\rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}(t)|m, n, e\rangle + b_{mn}(t)|m, n+2, g\rangle, \quad (20)$$

where the index  $m$  is referred to the motion and  $n$  to the field. If one finds  $a_{mn}(t)$  and  $b_{mn}(t)$ , the desired probability  $P_g(t)$  is just the summation in  $m$  and  $n$  of the squared absolute value of  $b_{mn}(t)$ . In the interaction picture, the coefficients obey the following system of differential equations

$$\begin{aligned} i\frac{d}{dt}a_{mn}(\tau) &= \chi_1 a_{mn}(\tau) + \chi_2 b_{mn}(\tau), \\ i\frac{d}{dt}b_{mn}(\tau) &= \chi_3 b_{mn}(\tau) + \chi_2 a_{mn}(\tau), \end{aligned} \quad (21)$$

where we defined

$$\begin{aligned} g &\equiv \frac{g_1 g_2}{\delta}; \quad \tau \equiv gt; \quad r = \frac{g_1}{g_2}; \quad \chi_1 \equiv \frac{f^2(m)}{r}(n+1); \\ \chi_2 &\equiv \sqrt{(n+1)(n+2)}f^2(m); \quad \chi_3 = rf^2(m)(n+2). \end{aligned} \quad (22)$$

The solution of (21) gives one all the information available about the physical system described by the Hamiltonian (15). For the purposes of this paper though, we are interested in the simplest case  $g_1 = g_2$  ( $r = 1$ ). Also, we will be particularly interested in initial preparations given either by

$$\begin{aligned} a_{mn}(0) &= e^{-|\alpha|^2/2} \frac{\alpha^m}{\sqrt{m!}} \delta_{np} \\ b_{mn}(0) &= 0 \end{aligned} \quad (23)$$

or

$$\begin{aligned} a_{mn}(0) &= e^{-(|\alpha|^2+|\beta|^2)/2} \frac{\alpha^m \beta^n}{\sqrt{m!n!}} \\ b_{mn}(0) &= 0. \end{aligned} \quad (24)$$

In both cases the ion is initially in the electronic excited state  $|e\rangle$ , and particularly in (23) the motion is in the coherent state  $|\alpha\rangle$  and the field in the Fock state  $|p\rangle$  containing exactly  $p$  photons, while in (24) the motion and field are initially prepared in coherent states  $|\alpha\rangle$  and  $|\beta\rangle$ ,

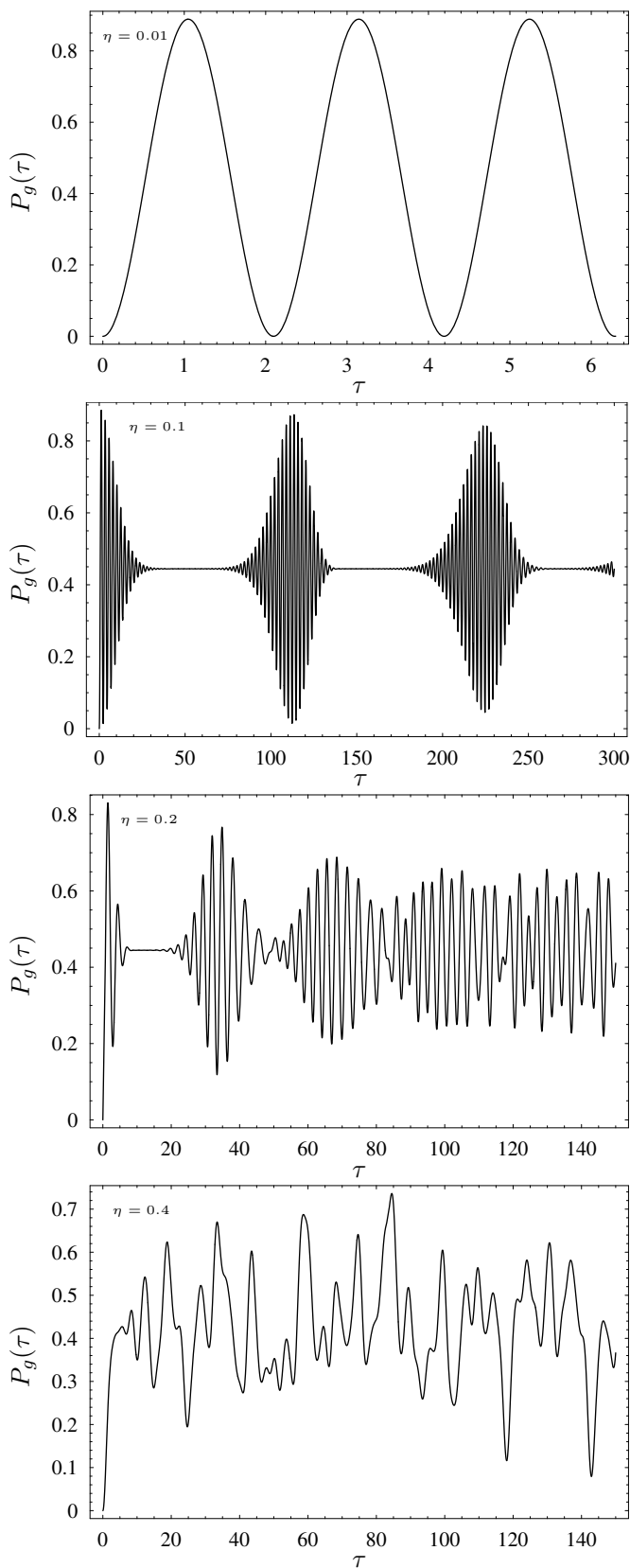
respectively. The experimental generation of vacuum Fock and coherent states of motion for trapped ions [8], as well as of electromagnetic cavity fields [19], have already been reported. The solution of (21) for  $r = 1$  and the ion initially in the excited state is

$$\begin{aligned} a_{mn}(\tau) &= a_{mn}(0) \left[ \cos(\Lambda_{mn}\tau/2) + i \frac{f^2(m)}{\Lambda_{mn}} \sin(\Lambda_{mn}\tau/2) \right] \\ b_{mn}(\tau) &= -2i a_{mn}(0) \frac{\chi_2}{\Lambda_{mn}} \sin(\Lambda_{mn}\tau/2), \end{aligned} \quad (25)$$

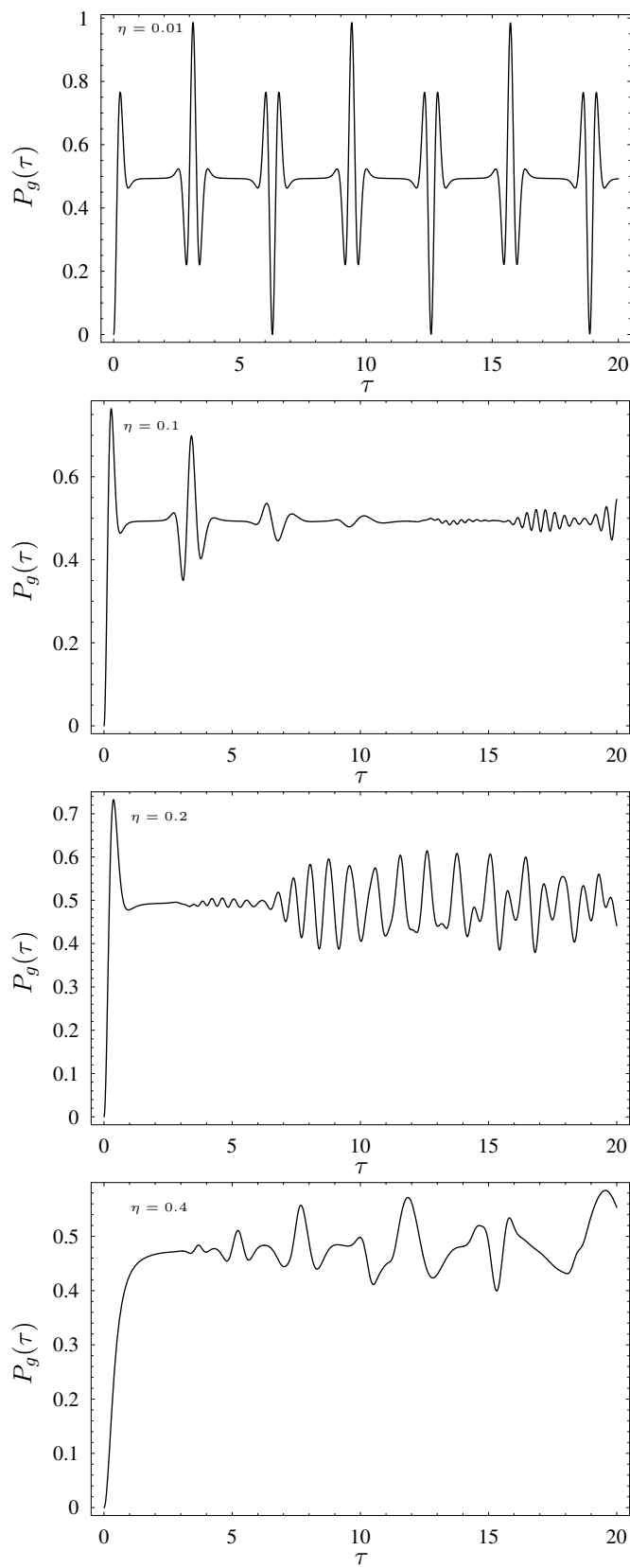
where  $\Lambda_{mn} \equiv \sqrt{f^4(m) + 4\chi_2^2}$ . The sought probability is finally found to be

$$P_g(\tau) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |a_{mn}(0)|^2 \frac{4\chi_2^2}{\Lambda_{mn}^2} \sin^2(\Lambda_{mn}\tau/2). \quad (26)$$

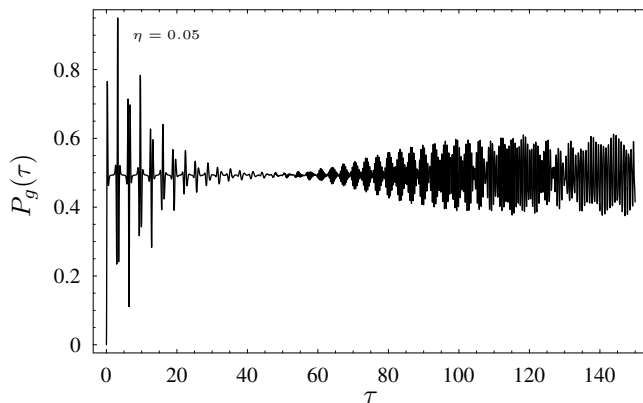
In what follows we will be studying the effect of the motion on  $P_g(\tau)$ . Mathematically, the ordinary TPJCM may be obtained by making  $\eta \rightarrow 0$ . We note that both the frequencies and amplitudes of each term in equation (26) have distinct contributions from the quantized field and from the ion's vibrational motion. For the first initial preparation (23), we clearly see [Fig. 2] how strong the influence of the atomic motion is because slight changes in the Lamb-Dicke parameter  $\eta$  lead to quite different behaviours. The expected Rabi oscillations found in the TPJCM are modified as the parameter  $\eta$  is increased. For small values of  $\eta$ , i.e. in the limit of the TPJCM, the coherent state of motion induces almost complete periodic collapses and revivals. By increasing  $\eta$ , the dynamics changes again and loses that periodicity until a complete irregular pattern takes place. Similar strong effects of the harmonic motion on the field dynamics in cavity QED setups have been already reported, see for instance Di Fidio et al. [13]. They consider a trapped ion in a Raman configuration interacting with a cavity field and an external laser. The main difference between our model and the one treated in [13] is that the two-photon configuration is in essence a periodic model while the Raman coupling does not necessarily lead to a periodic behaviour. Therefore, the inclusion of the atomic motion might not produce the same effects on those two different cavity-QED setups. In fact, the influence of the atomic motion should become more evident in the model treated here, as the oscillations induced by the atomic motion are in general not periodic, in contrast to the oscillations due to the interaction with the field. Considering now the second initial preparation (24), for an initial coherent state for the field, the dynamics is also modified by the harmonic motion [Fig. 3]. In this case, the characteristic (almost) periodic evolution found in the ordinary TPJCM and reported in several papers [6], is again modified by the increasing of  $\eta$ . The beats due to the statistics of the initial state of the center-of-mass motion (coherent state) clearly destroy the regular patterns. Moreover, for this initial preparation in which both the field and the center-of-mass motion of the ion are prepared in coherent states, we found for intermediate values of the LD parameter the interesting behaviour of revivals occurring at longer times [20], as shown in Figure 4. This *super*



**Fig. 2.** Time evolution of electronic ground state population for different values of the Lamb-Dicke parameter. The system has initially been prepared in the state  $|\psi(0)\rangle = |e, \alpha = 2, p = 0\rangle$ .



**Fig. 3.** Time evolution of electronic ground state population for different values of the Lamb-Dicke parameter. The system has initially been prepared in the state  $|\psi(0)\rangle = |e, \alpha = 2, \beta = 2\rangle$ .



**Fig. 4.** Long time evolution of the electronic ground state population. The system has initially been prepared in the state  $|\psi(0)\rangle = |e, \alpha = 2, \beta = 2\rangle$ .

*revival* or revival of revivals is a revival at long times of the Rabi oscillations and the ordinary short-time revivals. That is another special feature happening for this model that is not present in the TPJCM. Once the dynamics in the TPJCM is almost periodic for short and long interaction times, the concept of super revivals has no meaning in this case. All these different effects indicate some of the several interesting processes that might arise when considering cavity quantum electrodynamics with trapped ions, specially the model proposed here. We note, from the examples above, that the effects of the quantized field and the atomic motion are somehow superimposed and have peculiar features, such as, for instance, a dependence of the Stark shifts terms on the atomic motion.

### 3 Conclusions

We have investigated the interaction of a trapped ion with the quantized cavity field via two photon transitions. Particularly in this paper, we have studied this system focusing on the fundamental aspects of the understanding of light-matter interaction. We have derived an effective Hamiltonian under the rotating wave approximation and presented its analytical solution. This effective Hamiltonian contains motion-dependent Stark-shifts and a coupling constant which is a function of the intensity of motion, i.e. it is a function of the massive oscillator number operator. Both features are not present in the original two-photon Hamiltonian of the TPJCM. In particular, we have calculated the evolution of the population of the electronic levels, and found how the center-of-mass motion decisively changes the dynamics. In general, a treatment of the open system by including cavity losses, spontaneous electronic emission, and so forth, is necessary, although it rarely possesses a closed analytical solution. Some important and exceptional cases that admit analytical solutions for especial regimes were pointed out and treated by Di Fidio et al. [13]. Even though the system considered in [13] is not identical to ours as they treat the case of Raman transitions, most their findings may be applied

to our problem. As they point out, the inclusion of these realistic conditions will certainly have a destructive effect as long as quantum coherences are involved. In practice, some behaviours of  $P_g(\tau)$  presented here will not be observed in the current experimental setups, and it is the case for the super revival phenomenon. However, it is always important to investigate the presence or not of these typical quantum effects. On the other hand, differences between our model and the ordinary TPJCM arise also at shorter times as, indicated in Figures 2 and 3. Therefore, it would be possible to observe some of those effects with the achievement of the strong coupling regime; although losses and dephasing effects are present, they will have a much weaker magnitude than the cavity-ion coupling itself. The system comprising cavities and trapped ions studied here is now under intense experimental investigation with significant control improvements [21], and efforts are being made in order to reach the strong coupling regime.

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